Numerical Analysis of Stagnation Point Flows with Massive Blowing

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Introduction

WHEN the mass transfer from a surface either is independent or is only partially dependent on the rate of convective heating, e.g., when radiative heating is paramount, the rate of such mass transfer can be "massive." The boundary layer in this case involves a structure consisting of an inner layer dominated by pressure and inertial forces and of a thin, outer layer providing the adjustment of the inner layer to the inviscid, external flow. Our knowledge of laminar boundary layers under these circumstances has evolved as a result of a series of studies by several investigators (cf., e.g., Refs. 1–3).

Our present purpose is to provide the basis for the numerical treatment of the complex boundary layer at an axisymmetric stagnation point with massive blowing, where by "complex" we imply multicomponent, chemically reacting flows. It is known (cf., e.g., Ref. 4) that in these cases the usual methods for treating the problem of the two-point boundary conditions are either poorly convergent or nonconvergent and that it would be desirable to have an alternative, although approximate, method for obtaining solutions to the describing equations in these circumstances.† The failure of the aforementioned numerical methods as injection rates increase is due to the diminution of shear, heat transfer, and composition gradients near the wall and to the increasing extent of the boundary layer normal to the wall.

Our point of view is based on the assumptions that a satisfactory method exists for the solution of the complex boundary layer of interest for modest rates of mass transfer in terms of the usual similarity variable η and that it is desired to modify this method in a minimum fashion in order to obtain solutions for massive blowing. From this point of view the important contribution of Kubota and Fernandez,[2] although it provides a description of both layers of the boundary layer for arbitrary pressure gradient parameter β, is inapplicable because they found it convenient to carry out their analysis with the stream function as the independent variable. From the point of view cited previously it appears clearly advantageous to follow the asymptotic analysis, which is due to Kassoy,[5] which is employed by Libby and Kassoy[3] to study a three-dimensional boundary layer with massive blowing, and which deals directly with the boundary-layer equations in the similarity variable η. Although we concern ourselves here only with the case of dominant practical interest, namely the case of an axisymmetric stagnation point, the present analysis can be extended to cases for which $\beta \neq \frac{1}{2}$; the only complication which arises therein is that the solution for the velocity profile in the inner layer is given by a nonelementary function.

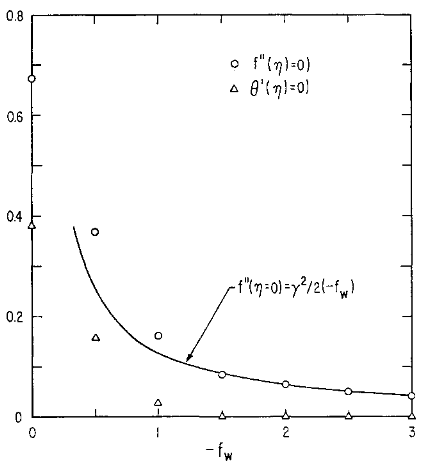

Fig. 1 Variation of wall parameters with mass transfer parameter.

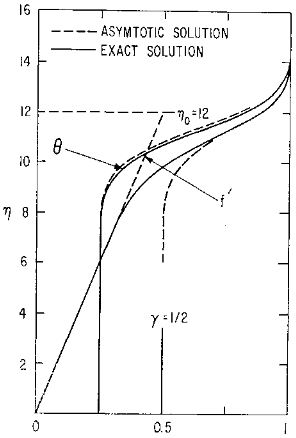

Fig. 2 Velocity and temperature profiles for $-f_w = 3$.

Analysis

In terms of the usual notation and subject only to the assumptions of a single diffusion coefficient‡ the equations describing the complex boundary layer at an axisymmetric stagnation point are (cf., e.g., Refs. 6 and 7)

$$(Cf'') + ff'' + (\tfrac{1}{2})[(\rho_e/\rho) - f'^2)] = 0 \tag{1}$$

$$[(C\bar{c}_p/S)\theta']' + \left[\bar{c}_p f + \sum_{i=1}^{N} (\bar{c}_{p,i}C/S)Y_i'\right]\theta' = \sum_{i=1}^{N} (h_i/c_{p,e}T_e)(2\alpha\rho_e)^{-1}(\rho_e/\rho)\dot{w}_i \tag{2}$$

$$[(C/S)Y_i']' + fY_i' = (2\alpha\rho_e)^{-1}(\rho_e/\rho)\dot{w}_i \tag{3}$$

$$i = I, \ldots N - I$$

where $\alpha = (du_e/dx)_{x=0}$. These equations are supplemented by equations for transport properties implied by C, S; by the thermodynamic properties implied by (ρ_e/ρ), $\bar{c}_p$, $\bar{c}_{p,i}$, $(h_i/c_{p,e}T_e)$; and by the chemical behaviour implied by $\dot{w}_i$. In addition appropriate boundary conditions are required; for our present, illustrative purposes we take the following, frequently employed conditions corresponding to injection through a porous surface: At $\eta = 0$:

$$f' = 0; f = f_w, \text{ given}; \theta = \theta_w, \text{ given}$$

$$(C/S)Y_i' = (-f_w)(Y_i - Y_{i,c}), \; Y_{i,c}, \text{ given}$$

$$\text{At } \eta \to \infty: f' = \theta = 1; \; Y_i = Y_{i,e}, \text{ given}$$

Now we are interested in cases of massive blowing so that $(-f_w) \gg 1$; following Refs. 2 and 3, we introduce as a small parameter $\epsilon \equiv (-f_w)^{-1}$, and seek in terms of Eqs. (1–3)

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‡ It will be seen from our results that the asymptotic treatment of multicomponent diffusion under conditions of massive blowing involves the numerical solution of free-mixing with such diffusion and thus that our analysis carries over to this more general case.

descriptions of the inner and outer layers which provide the structure of the boundary layer for $\epsilon \gg 1$. For the inner layer we introduce $\tilde{\eta} = \epsilon\eta, \tilde{f} = \epsilon f$; consider $\tilde{f}$ and θ and Y_i to be functions of $\tilde{\eta}$; and then assume a series in ϵ for each dependent variable. To lowest order we find that $\theta \simeq \theta_w$, $Y_i \simeq Y_{i,c}$, and that provided $\tilde{f} \neq 0$

$$\tilde{f}\tilde{f}''(\tilde{\eta}) + (\tfrac{1}{2})[\gamma^2 - (\tilde{f}')^2] = 0 \tag{4}$$

where $\gamma^2 \equiv (\rho_e/\rho_w)$ is known a priori by virtue of the features of the inner layer, which is approximately isothermal, and homogeneous with a composition associated with $Y_{i,c}$.§ The only boundary conditions which can be imposed on $f(\tilde{\eta})$ are

$$\tilde{f}(0) = -1, \tilde{f}'(0) = 0$$

so that the solution is

$$\begin{aligned} \tilde{f} &= -[1 - (\gamma\tilde{\eta}/2)^2] \\ \tilde{f}'(\tilde{\eta}) &= (\gamma^2/2)\tilde{\eta} \end{aligned} \tag{5}$$

Equation (5) gives the practical result

$$f''(\eta) \equiv f''(\eta = 0) \simeq (\gamma^2/2)(-f_w)^{-1}$$

But this solution is limited by the requirement that $\tilde{f} \neq 0$ to a finite range defining the inner layer, $0 < \tilde{\eta} < (2/\gamma) = \tilde{\eta}_0, 0 \leq \eta \leq 2(-f_w/\gamma) = \eta_0$.

Provided $\gamma \neq 1$, $\theta_w \neq 1$, and $Y_{i,c} \neq Y_{i,e}$ we clearly need an outer layer adjusting this inner layer to the external flow. Following Refs. 3 and 5, we employ a translational transformation; let

$$\begin{aligned} \hat{\eta} &= (\eta - \eta_0) - \lambda_1(\epsilon) - \dots \\ &= (\tilde{\eta} - \tilde{\eta}_0)\epsilon^{-1} - \lambda_1(\epsilon) - \dots, \end{aligned} \tag{6}$$

denote by $(\hat{\ })$ the dependent variables in the outer region, and assume a series in ϵ for each such variable. The lowest order terms are given by Eqs. (1–3) with $f \to \hat{f}$, $\theta \to \hat{\theta}$, $Y_i \to \hat{Y}_i$ and $\hat{\eta}$ the independent variable. Thus the equations describing the boundary layer for modest values of $(-f_w)$ also describe the outer layer for $(-f_w) \gg 1$, $(\epsilon \ll 1)$ with altered boundary conditions. For $\hat{\eta} \to -\infty$ we continue to have $\hat{f}'(\hat{\eta}) = \hat{\theta}(\hat{\eta}) = 1$, $\hat{Y}_i = Y_{i,e}$; we are permitted to let $\hat{f}(0) = 0$; but for the remaining conditions we must match the inner and outer solutions. With $\tilde{\eta} < \tilde{\eta}_0$ but fixed, $\epsilon \to 0$ implies $\hat{\eta} \to -\infty$ so that these additional conditions are imposed at $\hat{\eta} \to -\infty$. Thus the method of numerical analysis of Eqs. (1–3) we have assumed available for modest values of $(-f_w)$ must be altered to permit treatment of the three-point boundary value problem, namely to select $\hat{f}'(0)$, $\hat{f}''(0)$, $\hat{\theta}(0)$, $\hat{\theta}'(0)$, $Y_i(0)$, $Y_i'(0)$ $i = 1, \dots N - 1$ so that conditions at $\hat{\eta} \to \pm\infty$ are satisfied. Because these values at $\hat{\eta} = 0$ are not expected to be small and the range of $\hat{\eta}$ is not expected to be large, we might not anticipate any special numerical difficulties in successfully altering the existing methods. In any case, the outer solution is independent of $(-f_w)$ so it must be obtained only once for each set of parameters: θ_w, $Y_{i,c}$ $Y_{i,e}$, etc.

Matching leads to the following conditions at $\hat{\eta} \to -\infty$

$$\hat{f}'(\hat{\eta}) = \gamma, \hat{\theta} = \hat{\theta}_w, \hat{Y}_i = Y_{i,c}$$

$$\hat{f}(\hat{\eta}) = \gamma(\hat{\eta} + \lambda_1)$$

where λ_1 is a constant to be determined from this last equation.

Assume now that the solutions for the outer layer are available. We can construct a composite, first order solution to obtain

$$\begin{aligned} '(\eta) &= [\gamma^2/2(-f_w)]\eta + \hat{f}'[\hat{\eta} = \eta - 2(-f_w)\gamma^{-1} - \lambda_1] - \gamma \\ \theta(\eta) &= \hat{\theta}[\hat{\eta} = \eta - 2(-f_w)\gamma^{-1} - \lambda_1] \\ Y_i(\eta) &= \hat{Y}_i[\hat{\eta} = \eta - 2(-f_w)\gamma^{-1} - \lambda_1] \end{aligned} \tag{7}$$

With outer-layer solutions obtained numerically, Eqs. (7) provide approximate descriptions of the entire boundary layer. Usually an asymptotic analysis is employed to extend indefinitely, i.e., for $(-f_w) \to \infty$, full, numerical solutions carried out for a value of $(-f_w)$ sufficiently large so that within some desired accuracy the asymptotic solutions overlap the available numerical solutions. However, if as seems either likely or possible, the numerical analysis of the full problem of a complex boundary layer fails at some value of $(-f_w)$ not sufficiently large for Eqs. (7) to be applicable, there may be a gap of indefinite and indeterminable extent in the range of $(-f_w)$ before these solutions can be applied with assessable accuracy. Nevertheless Eqs. (7) may be useful for many engineering purposes. Of course additional terms in the asymptotic expansion can be readily calculated.

Numerical Example

To illustrate the principal features of this analysis we consider the simplified problem

$$\begin{gathered} f''' + ff'' + (\tfrac{1}{2})(\theta - f'^2) = 0 \\ \theta'' + f\theta' = 0 \\ f'(0) = 0, f(0) = f_w, \theta(0) = \theta_w = \gamma^2 = \tfrac{1}{4} \\ f'(\infty) = \epsilon(\infty) = 1 \end{gathered} \tag{8}$$

In Fig. 1 we compare the wall values $f''(\eta = 0)$, $\theta'(\eta = 0)$ corresponding to increasing $(-f_w)$ with the predictions of the asymptotic analysis and show that as far as these crucial wall values are concerned $(-f_w) \simeq 1.5$ appears large enough for these predictions to be quite accurate for this illustrative example.

The outer solution is found to yield $\lambda_1 = -0.5873$ so that the origin of the $\hat{\eta}$ variable in terms of the η variable is at $4(-f_w) - 0.5873$. We show in Fig. 2 the exact numerical solutions for the profiles for $(-f_w) = 3$ $(\epsilon = \tfrac{1}{3})$ and the inner and outer asymptotic solutions expressed in terms of the original η variable. It is clear that the asymptotic solutions give an accurate representation of the exact solutions. We repeat for emphasis that for the complex boundary layers providing the motivation for this Note it may not be possible to extend the exact calculations to the large values of $(-f_w)$ found possible in the illustrative example.

References

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§ We assume here that the gas mixture identified with the mass fractions $Y_{i,c}$ and with the temperature θ_w is nonreactive.